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NUMERICAL ANALYSIS

Abstracts of Presentations

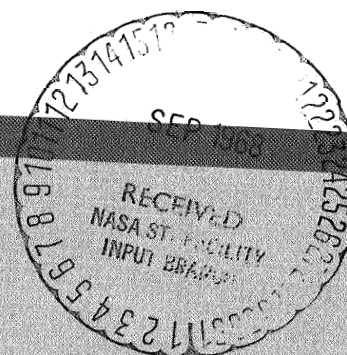
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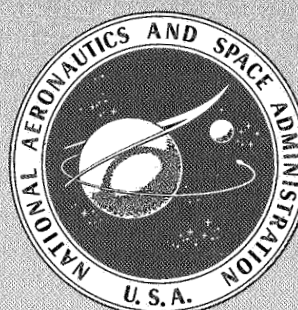
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A workshop held at
NATIONAL AERONAUTICS AND
SPACE ADMINISTRATION
Washington, D.C.
January 29-30, 1968



NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

NUMERICAL ANALYSIS

Abstracts of Presentations

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NASA Headquarters
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Edited by Raymond H. Wilson, Jr.



Scientific and Technical Information Division
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FOREWORD

A workshop meeting on numerical analysis was held at the National Aeronautics and Space Administration Headquarters, Washington, D.C., on January 29 and 30, 1968. Sixty persons, about half those invited as actively interested in numerical analysis applications to aerospace problems, registered attendance during the 4 sessions of the meeting, and of these, 34 contributed papers of which abstracts are printed here.

Nearly all of the contributed papers represent research done either at NASA centers, or elsewhere on contract or grant from NASA. Therefore, the spectrum of the collected abstracts shows many mathematical approaches to a wide range of aerospace problems, with perhaps the only essential characteristic being analysis directed toward ultimate numerical evaluation.

Of the many, besides the authors, who contributed to the success of this meeting, C. V. L. Smith (Atomic Energy Commission, Germantown, Md.), J. W. Siry (Goddard Space Flight Center, NASA), and L. C. Young (University of Wisconsin), who acted as chairmen of the last three sessions, deserve special thanks. Finally, it should be noted that in planning and other managerial details, the active collaboration of R. N. Seitz and others at Marshall Space Flight Center was practically indispensable. .

Raymond H. Wilson, Jr.
Chief, Applied Mathematics Program, NASA

NUMERICAL INTEGRATION OF LUNAR MOTION EMPLOYING A CONSISTENT SET OF CONSTANTS

Charles J. Devine, Jr.
Jet Propulsion Laboratory

A numerical integration and least-squares fit to the current JPL lunar ephemeris LE 4 (contained in JPL development ephemeris DE 19) over a 2-year period from April 25, 1966, to April 26, 1968, is described. The method of integration used and the differential equations solved, along with numerical methods employed to reduce computational error, are discussed. The fitted lunar ephemeris, when compared with Lunar Orbiter ranging data, reduced the residuals by several hundred meters.

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USE OF THE CONVOLUTION INTEGRAL FOR SIMULATION OF CONTINUOUS DYNAMICS

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A numerical method using the discrete convolution for the computation of all time responses of any complex multiloop control system is described in principle. The control systems are represented by block diagrams which are formed by the linear and nonlinear blocks well known in control engineering. The properties of these blocks are given by simple relations between input and output signals in the time domain at discrete and equal intervals. Any number of input signals may be fed into the system at any point of the system. The input signals and characteristics of the blocks may be arbitrary time functions given in analytical or numerical form.

For each linear block and each time interval the convolution must be calculated; this can be approximated by recursive formulas with a fixed number of only a few arithmetic operations. The results are then input to a system of algebraic equations which can be solved directly in the case of linear physical systems or iteratively in the general nonlinear case.

The method has the advantage that the block diagram can be directly mapped into a matrix equation where the position of the coefficients corresponds to the position of the blocks in the block diagram. There is no need for special treatment of algebraic loops and of sequencing. The mathematical method is especially suited for programming an electronic digital computer.

The method proved very powerful and fast compared with conventional methods of solving ordinary differential equations in a number of applications.

It is felt that the convolution method deserves more attention by numerical mathematicians in order to make optimum use of it and to explore its theoretical limitations.

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THE CALCULATION OF ATMOSPHERIC PROPERTIES

Wm. A. Mersman
Ames Research Center, NASA

Current methods of computing the properties of the Earth's atmosphere are quite efficient for altitudes less than 90 kilometers. For higher altitudes the methods are intolerably slow, primarily because of the difficulty encountered in computing the pressure. This requires the integration of a rational fraction of high degree, based on the exact formula for geopotential altitude in terms of geometric altitude.

A simple, linear rational fraction for geopotential altitude as a function of geometric altitude has been obtained by combining analytical and numerical techniques. Use of this approximate formula provides an algorithm for calculating the pressure with a speed that is essentially independent of altitude. Thus, above 90 kilometers, the new method is about 12 times as fast as current methods, with no loss in accuracy.

THE SYSTEM OF SINGULAR INTEGRAL AND ORDINARY DIFFERENTIAL EQUATIONS FOR THE COMBINED RADIATION CONDUCTION HEAT TRANSFER OF A FINNED SPACECRAFT CYLINDER

Edward M. Keberle
Jet Propulsion Laboratory

The computation of the equilibrium temperature distribution function of a finned cylinder in the radiation field of the Sun takes into account the following energy transfers at each infinitesimal surface element:

- (1) Absorption and reflection of incident radiation from the Sun or other parts of the finned cylinder
- (2) Emittance of radiation from the surface according to the relationship $W = \epsilon \sigma T^4$
- (3) Conduction of heat inside the surface to neighboring surface elements.

These requirements lead to the following system of equations in terms of the unknown temperature $T(x)$ as well as of the unknown radiosities $B(x)$ resp. $C(x)$ of the outer resp. inner parts of the cylinder:

$$\frac{d^2 T(x)}{dx^2} + \frac{a_1}{x} \frac{dT(x)}{dx} + a_2 T^4(x) + B(x) + a_4 C(x) + a_5 = 0$$

$$B(x) + \beta_1 T^4(x) + \beta_2 \int_0^{l_1+l_2+l_3} K_{\text{outer}}(x,y) B(y) dy + \beta_3 = 0$$

$$C(x) + \gamma_1 T^4(x) + \gamma_2 \int_{l_1}^{l_1+l_2} K_{\text{inner}}(x,y) C(y) dy + \gamma_3 = 0$$

The mathematical complexities consist of –

- (1) A combined system of nonlinear ordinary differential and nonlinear integral equations
- (2) Singular kernel
- (3) Nonlinear boundary conditions
- (4) Discontinuous solution $B(x), C(x)$

Such a problem cannot be solved either analytically or by the construction of particular subroutines. Relevant is the applied mathematical and numerical task of adjusting common machine subroutines to the full realistic problem, which is filled with traps and exotic requirements.

A variant of the method of successive approximations was used to construct $B_n(x), C_n(x)$ by means of the integral equations while keeping $T_{n-1}(x)$ fixed and then solving the boundary value problem of the differential equation to obtain $T_n(x)$, while $B_n(x), C_n(x)$ remain unchanged.

Convergence was improved and accuracy of 0.1 percent obtained by internal repetition of the differential equation routine in comparison with the integral equation routine.

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THE AUTOMATIC INTEGRATION OF STIFF ORDINARY DIFFERENTIAL EQUATIONS*

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Stiff differential equations arise in physical systems in which two or more time constants are greatly different. They can arise also when a single time constant is very small in relation to a forcing function. **An** example of the latter is the equation $y' = \lambda(y - F(t)) + F'(t)$, which has the solution $y = ce^{\lambda t} + F(t)$. If $\lambda \ll 0$ and $F(t)$ is a smooth function, the quantity $ce^{\lambda t}$ is negligible after a short time period. Thereafter, it would be desirable to use step sizes h that control the error due to $F(t)$. Unfortunately, the stability of the numerical methods in current use requires that $|h\lambda|$ be bounded (typically it must be less than 1). We now have multistep methods that are stiffly stable in the sense that they are stable for **all** $\text{Re}(h\lambda) < D < 0$ for some D (ref. 1). These methods have allowed the step size to be increased by several orders of magnitude over the conventional methods such as Adams.

A class of multistep methods are used for the integration (refs. 2 and 3). These references show estimates of the various derivatives available. The difference between the predictor and the corrector is also used to estimate higher derivatives. These estimates make it possible to change the step size and the order so that a minimization of the amount of work per step might be attempted. An additional advantage of automatic order selection is that the initial order can be set to one. Because it is then a one-step method, no special starting procedures are required. Comparisons of these methods with Adams' methods have been made for nonstiff, moderately stiff, and very stiff equations (ref. 4). The numerical results indicate that Adams' method is better than the stiff method by a factor of less than 2 for nonstiff equations, but very much inferior for stiff equations. The automatic order methods are superior to the fixed order methods for the equations considered.

It is concluded that stiff methods should probably be used if the question of stiffness is open. If the user can be certain that his equations are not stiff, an Adams-type method is better. One unfortunate feature of the method is that the partial derivatives $\partial y_i / \partial y_j$ are needed approximately. To save the user the work of preparing subroutines for both $y'(y)$ and $\partial y'(y) / \partial y$, a package is being constructed which will accept the differential equations in symbolic form and compile 360 codes for the functions $y'(y)$ and their partial derivatives. It is proposed to include the Adams' and the stiff methods in the final package so that the choice of method, order, and step can be made.

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SERIES REPRESENTATIONS FOR GENERALIZED TEMPERATURES

Deborah Tepper Haimo
Southern Illinois University

The generalized heat equation is given by

$$\Delta_x u(x,t) = \frac{\partial}{\partial x} u(x,t) \quad (1)$$

where $\Delta_x f(x) = f''(x) + \frac{2\nu}{x} f'(x)$, and ν is a fixed positive number. A generalized temperature is defined as a C^2 solution of equation (1). The fundamental solution of equation (1) is the function $G(x;t) = G(x,0;t)$ where

$$G(x,y;t) = \frac{1}{(2t)^{\nu+1/2}} e^{-(x^2+y^2)/4t} \Omega \frac{xy}{2t}$$

with $\Omega(z) = 2^{\nu-1/2} \Gamma(\nu+1/2) z^{1/2-\nu} I_{\nu-1/2}(z)$, $I_a(z)$ being the ordinary Bessel function of imaginary argument of order a . The solution of the Cauchy problem of equation (1) with $u(x,0) = x^{2n}$ is the polynomial

$$P_{n,\nu}(x,t) = \sum_{k=0}^n 2^{2k} \binom{n}{k} \frac{\Gamma(\nu+1/2+n)}{\Gamma(\nu+1/2+n-k)} x^{2n-2k} t^k$$

The Appell transform of $P_{n,\nu}(x,t)$ is the function

$$W_{n,\nu}(x,t) = G(x;t) P_{n,\nu}\left(\frac{x}{t} - \frac{1}{t}\right)$$

A function $u(x,t)$ belongs to the class H^* for $a < t < b$ if and only if it is a generalized temperature satisfying the condition that for all $t, t', a < t' < t < b$,

$$u(x,t) = \int_0^\infty G(x,y;t-t') u(y,t') d\mu(y), \quad d\mu(y) = [2^{\nu-1/2} \Gamma(\nu+1/2)]^{-1} y^{2\nu} dy,$$

the integral converging absolutely for all x . Note that the polynomial $P_{n,\nu}(x,t)$ belongs to H^* for $-\infty < t < \infty$, and its Appell transform $W_{n,\nu}(x,t) \in H^*$ for $t > 0$.

Theorem: A necessary and sufficient condition that

$$u(x,t) = \sum_{n=0}^{\infty} a_n P_{n,\nu}(x,t)$$

for $|t| < \sigma$ is that $u(x,t) \in H^*$ for $|t| < \sigma$. The coefficients a_n are given by either

$$a_n = \frac{u^{(2n)}(0,0)}{(2n)!}$$

or

$$a_n = \frac{\Gamma(v+1/2)}{2^{4n} n! \Gamma(v+1/2+n)} \int_0^\infty W_{n,v}(x,t) u(x,-t) d\mu(x), \quad 0 < t < \sigma.$$

Theorem: A necessary and sufficient condition that

$$u(x,t) = \sum_{n=0}^{\infty} b_n W_{n,v}(x,t)$$

for $t > \sigma \geq 0$ is that there exist a function $v(x,t) \in H^*$ for $|t| < 1/\sigma$ such that $u(x,t)$ will be its Appell transform. Here $b_n = v^{(2n)}(0,0)/(2n)!$

The theory for the representation of generalized temperature functions in series of polynomial solutions and of their Appell transforms is analogous to that for the expansions of analytic functions in series of monomials $p_n(x) = x^n$ and of inverse monomials $w_n = 1/x^{n+1}$, respectively.

d

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APPLICATION OF TWO TECHNIQUES IN THERMAL RADIATION PROBLEMS

P. A. Newman and R. W. Barnwell
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The two techniques which are applied to thermal radiation problems are Lax's finite difference method and Case's singular normal mode expansion method. Lax's method is a numerical technique for solving nonlinear hyperbolic equations, whereas Case's method is basically a rigorous mathematical technique for solving the linear transport equation. Neither of these techniques is new, and they are not applied to the same type of radiation problem. However, applications involving thermal radiation and emission are recent and are currently being investigated.

Specifically, Lax's finite difference technique is used to solve the problem of nonadiabatic flow of a thermally radiating gas past a sphere. The solution of the steady-state flow problem is sought as the asymptotic solution of the time-dependent flow problem; this method has been used in the past for solving steady-state flow problems in this way. However, when the gas is allowed to absorb and emit radiation, integral terms appear in the energy equation. It is shown that these can be treated in a straightforward manner, since the complete solution is known on time planes previous to the one on which the solution is being sought. Therefore, integrals across the flow field can be evaluated.

The second technique was developed by Case to solve one-speed, one-dimensional neutron transport problems. Basically, he showed that a separation of variables in the homogeneous linear transport equation could be performed and that the resulting eigensolutions, although singular, were complete and orthogonal. Thus, any other solution can be expanded in terms of them, the expansion coefficients being determined by the source and boundary conditions of the particular problem. Application of Case's technique in radiative transfer is illustrated by solving the problem of a finite slab of gray gas between infinite parallel plates. The use of this technique allows one to obtain accurate solutions with very little numerical analysis.

BAIRSTOWS METHOD REVISITED

R. W. Hamming
Bell Telephone Laboratories, Inc.

The basic purpose of this report is to point out the desirability of designing library routines to meet the use that is going to be made of the answers, rather than merely meeting the obvious mathematical criteria. Bairstow's method for finding zeros of polynomials is used as an example.

Among other points made is that a routine which reports close distinct zeros will usually cause severe roundoff problems much later in the computation, whereas if they are reported in a multiple zero then the roundoff will not occur.

The paper discusses methods of overcoming some of the well-known faults of Bairstow's method and identification of multiple zeros. Simplicity in understanding what is going on is also emphasized.

ANALYTIC APPROXIMATIONS TO THE SOLUTIONS OF VARIATIONAL EQUATIONS

Samuel Pines
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The solutions of the differential variational equations of a dynamical system are often required in problems involving gradient search methods, two-point boundary value problems, optimal control theory, differential correction schemes, and so forth. Usually this is accomplished by numerical integration at the cost of an increase in complexity and computing time. Since, for many search problems, the exact value of the partial derivative is not specifically required, a method is presented for obtaining an approximation to the solution of the variational problem by analytically differentiating a known closed-form solution of an approximating dynamical system.

Specific examples are described, such as the closed-form solution of the two-body problem for perturbation problems in orbital mechanics and the known solution of the motion of a vehicle in a uniform gravitational field under the action of an inertially fixed thrust of constant magnitude for use in error propagation of vehicles under thrust.

QUASILINEAR FORMULATION OF NONLINEAR TRANSFORMATIONS

Henry Wolf

Analytical Mechanics Associates, Inc.

In many computations, a set of parameters has to be changed. Generally, two types of transformation formulas are used—one for infinitesimal and one for finite changes. The infinitesimal transformation is usually linearized and behaves properly when the changes tend to zero, but behavior becomes improper when the changes become finite. The finite transformations, on the other hand, do not properly tend to zero when the transformation is small. The boundary between the two regimes where each transformation is valid is usually ill defined. Further, transformations often depend on several parameters, some infinitesimal and some finite. In this case, neither formula is completely satisfactory.

In the paper a method is presented to obtain a “quasilinear” form, which is valid for finite changes, but contains the nonlinearities as a coefficient of the transformation parameters, and consequently behaves properly also for the infinitesimal case.

Applications of this method are presented. They include quadratic Newton-Raphson iteration, the planetary and Encke perturbations in equations of motion, vector rotation formulas, and differential correction schemes.

IS THERE A UNIFIED SOLUTION FOR A CUBIC EQUATION?

Mary H. Payne

Electronics Research Center, NASA

A method of quickly and accurately solving the cubic equation $ax^3 + bx^2 + cx + d = 0$ is desired for various combinations of “largeness” and “smallness” of the coefficients. If, for example, a is not too small, the standard Cardan method is acceptable. If a is small but d is not, one can divide by x^3 and solve for $1/x$. If $3ax^2 + 2bx + c$ is not too small, Newton’s method can be used. The question posed is, How does one formulate a sequence of tests on the coefficients together with methods of solution for various outcomes of these tests so as to cover **all** cases? Suggestions made by Gear (University of Illinois) and Hamming (Bell Telephone Laboratories) are currently under study.

THE NUMERICAL SOLUTION OF SINGULAR INTEGRAL EQUATIONS OF POTENTIAL THEORY

M. S. Lynn and W. P. Timlake
IBM Houston Scientific Center

The authors discuss integral equations that arise out of three-dimensional potential theory; i.e., where the kernel is*

$$K(P,Q) = -\frac{1}{2\pi} \frac{\partial}{\partial n_p} \left(\frac{1}{|r_{PQ}|} \right)$$

Equations of this type are singular both in the sense of the kernel being singular [$\sigma(1/r)$] as well as the operator being singular. The latter singularity is deflated out by seeking that particular solution φ for which

$$\int_S \varphi dA = 0.$$

The surface S is “triangulated” into n areal elements of maximum diameter ρ . A numerical scheme derives a system of approximating linear algebraic equations of order n in ψ .

Based on the assumption, inter alia, that the surface S is regular in the sense of Kellogg and Lyapunov, it is shown that ψ differs from the average value of φ in the L_∞ norm by at most $K\rho^{\beta+1/3}$ (K is a constant). The algebraic equations are solved by Jacobi iteration, in which the spectral radius is bounded (as $n \rightarrow \infty$) strictly away from 1; for example, for the model problem, a sphere, the spectral radius is of order $1/2$. Extensions to multiconnect regions were also considered.

*For notation, precise statements, and a proper bibliography, see a paper with the same title which is to appear shortly in *Numerische Mathematik*.

SOLUTION OF SPARSE LINEAR SYSTEM OF EQUATIONS

R. P. Tewarson

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The problem of transforming a given sparse matrix A into a block diagonal form (bdf) and the subsequent transformation of each of the block diagonal matrices into as nearly upper triangular form (utf) as possible, by using only row and column permutations, is discussed. It is shown how some of the results from graph theory can be used to transform A to the bdf. To transform the block diagonal matrices into the utf's, two methods are described, one of which uses linear programming while the other uses approximate probabilistic arguments. The latter method, in relation to the computational effort, yields significant results in practice.

AUTOMATIC ERROR BOUNDS FOR REAL ROOTS OF POLYNOMIALS HAVING INTERVAL COEFFICIENTS

Richard J. Hanson
Jet Propulsion Laboratory

The notion of interval arithmetic (ref. 1) is briefly described, and a Newton-Raphson algorithm using interval arithmetic and range extensions of functions is discussed (ref. 2). **This** algorithm is particularly useful for finding real roots of polynomials with interval coefficients, and several examples are presented.

The discussion is concluded with a presentation, and example, of an algorithm for solving a square linear system of algebraic equations of full rank (ref. 3).

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NONLINEAR PROBLEMS AND MATHEMATICAL METHODS IN AEROELASTICITY

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Princeton University

A discussion of methods for mathematically treating the dynamic interaction of a flowing fluid and flexible solid (aeroelasticity) is given. Primary emphasis is placed upon techniques for handling the nonlinear behavior of such physical systems. The choice of numerical procedures is based upon the great ability of high-speed digital computers to numerically integrate systems of ordinary integral-differential equations (initial value problem). Hence, the nonlinear partial differential field equations governing the fluid and solid motions are reduced to such systems by utilizing generalized Fourier or eigenfunction expansions associated with the linear dynamics of the solid. Given these expansions, the fluid motion is determined from the fluid field equations, through the methods of the transform calculus, in terms of the (as yet unknown) time history of the solid motion. The fluid forces on the solid are then calculated, and the system of ordinary (nonlinear integral-differential) equations for the solid motion are subsequently determined from the solid field equations. Numerical integration of these ordinary integral-differential equations gives the time history of the solid motion. The flow field time history also may be determined if desired. Finally, it should be mentioned that the Fourier or eigenfunction expansions for the linear solid which are needed may be determined analytically for simple solid geometries; however, for irregular geometries, the eigenfunctions must be calculated by yet another numerical procedure, e.g., the finite difference method. Less frequently, this also may be true of the flow field. It is emphasized that finite difference methods applied directly to the full nonlinear, fluid-solid interaction problem are impractical from the point of view of computer time and storage requirements.

REGULARIZATION AND STABILIZATION OF NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Yale University

The transformations which regularize the equations of motion of celestial mechanics involve the independent and the dependent variables. The purpose of these transformations is to eliminate the singularities occurring at collisions of two bodies; however, it is well known that even when collisions do not occur, the appearance of close approaches makes regularization mandatory from the point of view of numerical analysis. It was found that these regularizing transformations offer improved stability characteristics of the equations of celestial mechanics.

Examples regarding the restricted problem of three bodies as applied to Earth-to-Moon trajectories show a significant time saving on electronic computers when regularizing variables are used. In comparison studies, the methods of regularization show advantages over methods which do not eliminate the singularities in accuracy and in efficiency. Analytical approaches attempting to arrive at series solutions to Apollo-type trajectories also benefit from the process of regularization.

Another example is the problem of integrating systems of ordinary differential equations of high order, say several hundred. Such n-body integrations show sensitivity and sudden increases in the error at any time when close approaches of the participating bodies occur.

The final example demonstrating the importance of regularizing transformations is the solution of a problem of three bodies proposed by Burrau in 1913 and solved only recently. The repeated close approaches of the three participating bodies necessitate the use of regularizing variables originally proposed by Levi-Civita and applied to Burrau's problem by Szebehely and Peters.

REFERENCE

Szebehely and Peters: *Astron. J.*, vol. 72, Sept. 1967, pp. 876-883.

SUBARC ELIMINATION AND OPTIMAL ORBIT TRANSFER

Thomas L. Vincent
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A procedure for handling optimization problems with corners resulting from the imposition of restrictions is presented. The requirements which must be met in order to apply the methods presented are that the dynamical equations of constraint contain only one control variable and that these equations can be analytically integrated along any segment of the trajectory in which the control law is restricted. Under such circumstances it is shown that the restricted segments of the optimal trajectory may be effectively eliminated in the process of determining the changes in the Lagrange multipliers, state variables, and Hamiltonian function from one side of the restricted segment to the other. A set of conditions which may be used to determine the changes or "jumps" in the Lagrange multipliers and Hamiltonian function are obtained directly from a set of corner conditions developed for trajectories with discontinuities in the state variables. With the use of these conditions, there is no need to integrate the Euler equations along the restricted arc, as is usually done in the literature.

As an example as to where these methods may be directly applied, the thrust-coast-thrust optimal orbit transfer problem is cited. It is shown that the optimal cutoff and relight points for the thrust vector could be obtained directly from the corner conditions, rather than by integration of the Euler equations along the restricted arc.

REFERENCE

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A SEGMENTAL EXTENSION OF SOLUTIONS TO OBTAIN A NUMERICALLY STABLE BOUNDARY VALUE PROBLEM SOLUTION

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A technique is presented to circumvent the possible loss in accuracy associated with the superposition of initial value-problem solutions to obtain the solution to a specified boundary value problem. Should the dependent functions Y_i of the initial value problems exhibit a rapid growth with increasing independent variable x , then serious cancellation of leading digits could result upon superposition to meet the remote edge boundary conditions.

The suggested technique entails a repeated use of superposition of initial value problems at small segmental intervals in x in order to suppress the offending growth in the functions Y_i . Specifically, at each segmental station x_i , initial value problem solutions are combined, first, to meet the desired remote edge boundary conditions artificially at x_n , and again to provide the required independent complementary solutions for continuing through the next segment. The resulting solutions then provide sets of initial conditions for extending the integration over the next segment to x_{i+1} , where the superposition procedures are repeated. The process is iterated until, at the remote edge x_n , the boundary conditions are satisfied through the final segmental superposition of solutions. By keeping the segment size suitably small, the growth in magnitude of the dependent functions is accordingly controlled, and acceptable error is introduced at each superposition operation.

VARIABLE-MESH, MULTISTEP METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS"

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A theory for variable-mesh, multistep, predictor-corrector solution of ordinary differential equations is presented. The difference equations employed are generalizations, for the case of variable mesh spacing, of previous formulas requiring fixed step size. In addition to retaining the high local accuracy and convergence properties of the earlier methods, the variable mesh method is developed in a form conducive to the generation of effective criteria for the selection of subsequent step sizes in the step-by-step solution of differential equations. These criteria are based on considerations of truncation error, convergence of corrector iterations, and an extensive treatment of relative numerical stability. This treatment included tracking the roots of a cubic equation, whose coefficients are complex in the case of simultaneous differential equations, and determining stability as a function of one real and one complex parameter. The algorithm has been tested extensively and compared with other methods. The results of the comparison favor the new method.

Current research on the extension of the variable mesh method to systems whose components decay at greatly different rates is also presented.

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STEP-SIZE SELECTORS FOR THE SHANKS FORMULAS OF RUNGE-KUTTA TYPE

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In 1962 and later, the author* developed the only Runge-Kutta-type formulas in existence of orders 7, 8, 9, and 10. In addition, more efficient formulas were derived using 4, 5, 6, 7, and 10 evaluations. The 9th- and 10th-order formulas were developed in 1967. To indicate the complexity involved, it is noted that the 10th-order formula is derived by solving considerably more than 500 algebraic equations, more than half of which are of the 18th degree.

A Computation Division report from Marshall Space Flight Center, NASA, states that Shanks' formulas through the 8th order require only about one-third as much machine computation in orbital and interplanetary probes as the previous Runge-Kutta-type formulas.

Recently, the author has discovered that a second formula, less accurate but using no new evaluations, can be developed for each of the Shanks formulas. Let us indicate the approximations to y by Y_1 and Y_2 , the errors by e_1 and e_2 , and the evaluations of f by f_i so that $y_0 + h \sum c_i f_i = Y_1 = y + e_1$, where y_0 is the initial value of y and c_i are constants of the formula. Subtracting Y_2 , we get $h \sum (c_i - d_i) f_i = Y_1 - Y_2 = e_1 - e_2$. The quantity $S = \left| h \sum (c_i - d_i) f_i \right|$ is the absolute value of a fairly good approximation to the error e_1 of the original formula. It is called the step-size selector and can be used to change the step size for the next step of the integration process (if necessary) so that

$$10^{-n} \leq S \leq 10^{-n+4}$$

for an appropriate integer n . Other intervals for S may be used. Tests indicate that, in order to maintain the same accuracy, formulas using the selectors will require only about one-third the computing time used for the original formula alone.

These innovations are programed and in use at the Computation Division of Marshall Space Flight Center.

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RATIONAL APPROXIMATION OF SPECIAL FUNCTIONS

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Subroutines based on the best (minimum-maximum relative error) rational approximations are ordinarily the most efficient for computing mathematical functions. For continuous functions of a single real variable, several Fortran and Algol programs are available which can produce rational functions differing from the best approximation by no more than 0.1 percent in less than 1 minute on even a moderate-speed computer. The major requirement for using these programs is a reference routine producing values of the desired function to three to four digits more than will be required for the approximation. Typically, the resulting approximations yield up to 8 significant digits for the ratio of 2 cubics, and up to 20 for the ratio of 2 sextics.

Aside from constructing and checking the reference routine, the principal difficulties are the choice of the particular auxiliary function to be approximated for various portions of the domain, and the determination of the intervals over which each form is to be used.

J. F. Hart et al. show the best rational approximations ranging up to 25S for the elementary, and several higher functions, including the error function, the gamma function, the complete elliptic integrals, and the Bessel functions $J_0(x)$ and $J_1(x)$. It also gives an extensive bibliography of other published approximations.

Several other groups are also active in approximating additional functions. We would be interested in learning of other functions which are of sufficient importance in applications to justify their approximation.

REFERENCE

Hart, J. F., et al.: Computer Approximations. John Wiley & Sons, Inc., 1968.

EXTENSIONS AND APPLICATIONS OF THE HOUSEHOLDER ALGORITHM FOR SOLVING LINEAR LEAST-SQUARES PROBLEMS

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Let A denote a real matrix of rank r having m rows and n columns and let \vec{b} denote a real vector having m elements. The problem of determining an m -vector \vec{x} which exactly or approximately satisfies $A\vec{x}=\vec{b}$ occurs in many contexts. We distinguish three cases: (1) $m=n$, (2) $m>n$, and (3) $m<n$, and further subdivide each case into two subcases: a, $r=\min(m,n)$ and b, $r<\min(m,n)$.

In case (1)a the matrix A is square and nonsingular. In the five other cases it is necessary to produce a square $r \times r$ matrix of rank r at some stage of the computation and solve a system of equations involving that matrix. For example, the commonly used method of solving case (2)a in the sense of least squares involves the formation of the $r \times r$ matrix $A^T A$. Similarly, a standard method for solving case (3)a uses the $r \times r$ matrix AA^T .

It is known that the spectral condition number of $A^T A$ (or of AA^T) is the square of the condition number of A . It is also known that there are methods using orthogonal transformations which permit the reduction of A to an $r \times r$ matrix whose condition number is the same as that of A . Because of the conservation of the condition number, these latter methods are, in general, capable of preserving more accuracy in the solution than the methods using $A^T A$ or AA^T , assuming the same type (i.e., precision) of arithmetic is used. More specifically, a computation which can be carried out satisfactorily with orthogonal methods using s -digit arithmetic may require $2s$ -digit arithmetic, if done using $A^T A$ or AA^T .

In some, possibly most, cases of badly conditioned systems of linear equations, it is more meaningful to compute a solution of an appropriate neighboring system of lower rank that is reasonably well conditioned than to compute an accurate solution for the badly conditioned problem. Certain orthogonal methods adapt rather naturally to this approach.

Computer users will be better served as subroutines using orthogonal methods become more readily available and these methods are more widely understood by people who specify and design computer programs. For example, reference 1 illustrates how a user's time can be absorbed in an essentially random search for a reliable least-squares program. Fortunately, one of the programs this user encountered happened to use orthogonal methods and, indeed, he did find that it was the only satisfactory program for his test cases.

One algorithm for solving problems (1)a and (2)a by orthogonal methods, specifically by the use of Householder orthogonal transformations, is given in reference 2. In an effort to improve the flexibility and accessibility of orthogonal methods for general applications, we have extended the algorithm of reference 2 to provide a uniform approach to problems (1)a, (2)a, and (3)a. With a somewhat arbitrary criterion for rank estimation, this algorithm also encompasses problems (1)b, (2)b, and (3)b. This algorithm permits sequential processing of the rows of the augmented matrix $[A : \vec{b}]$ so that the storage required depends only on n and not on m . This algorithm has been implemented as a Fortran IV subroutine.

A different use of orthogonal methods is the singular value decomposition algorithm of reference 3. We have converted the Algol code given in reference 3 to Fortran IV. This provides another numerically sound way to obtain the capabilities mentioned in the preceding paragraph.

It is expected that the ability to produce a numerically sound solution for $A\vec{x} = \vec{b}$ regardless of the relationship between m , n , and r will be valuable in many applications. We are investigating the use of these methods in the solution of nonlinear equations, nonlinear least squares, linear and non-linear l_p minimization ($1 \leq p \leq \infty$), computation of independent eigenvectors associated with multiple eigenvalues, root locus computations, implicit function evaluation, minimization subject to constraints, and solution of large systems of equations.

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3. Golub, Gene H.; and Businger, Peter: Least Squares, Singular Values, and Matrix Approximations With an ALGOL Procedure for Computing the Singular Value Decomposition. Tech. Rept. CS73, Computer Science Department, Stanford Univ., July 1967.

TRUNCATION ERRORS ON THE COMPUTATION OF ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS BY GAUSS AND LANDEN TRANSFORMATIONS

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It is well known that algorithms based on quadratic transformations yield a high rate of convergence in the computation of elliptic integrals, which occur frequently in astronomical and astronautical problems, such as in artificial satellite theory. Four algorithms based on descending and ascending Gauss and Landen transformations have been investigated quite thoroughly, either in the trigonometric or algebraic versions, the latter commonly known as the **AGM** scale (ref. 1).

In actual computer application, these algorithms present themselves in some form of an infinite iterative process, to be truncated after a finite number of steps. Some bounds and estimates obtained for errors involved in such truncation are reported. Such error analysis is to serve a twofold purpose: first, as a theoretical guide before computer programming, and second, to give actual numerical value of the errors during computer programming. A small tolerance parameter is chosen consistent with both the algebraic and trigonometric versions and applicable in all four algorithms. Error bounds in terms of this parameter are given for the computation of elliptic integrals of the first and second kinds.

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2. Ng, E. W.: Truncation Errors on the Computation of Elliptic Integrals of the First and Second Kinds by Gauss and Landen Transformation. To be published.

APPLICATION OF INTEGRAL EQUATION METHOD TO NUMERICAL ANALYSIS OF BIHARMONIC PROBLEMS

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This presentation is based on reference 1 and on subsequent work which is to be published shortly. The method of analysis is based on a general formulation of a biharmonic stress function in terms of two harmonic functions. The harmonic functions are each represented by single-layer potentials, each containing an unknown source density function. Then the boundary conditions yield two coupled integral equations closely related to those of the Fredholm type. Numerical solutions for the unknown source densities are obtained by assuming that the source densities are piecewise constant around the boundary of the domain, thereby replacing the integral equations by sets of simultaneous algebraic equations which can be effectively solved by using a high-speed digital computer. Once the source density functions are determined, stress in the elastic domain may be found by computing second derivatives of the stress function.

Numerical results obtained by the author, as well as other investigators, indicate that the integral equation method is extremely accurate at a small distance away from the boundary, but that, in general, the accuracy of the numerical solution decreases as the point at which computations are performed approaches the boundary. Improvement of the accuracy of numerical results on and near the boundary is therefore identified as the central problem of the integral equation method.

In the previous report (NASA CR-779), it was first shown that one of the three quantities to be computed at the boundary, namely, the total stress, can be evaluated directly from the approximate stress function. The research reported herein shows that greater accuracy in computed boundary stress components can be obtained from the numerical solution if the numerically determined, piecewise constant source density functions are replaced by functions which, with their first derivatives, are continuous on the boundary. In contrast to other reported results, this modification permits one to compute unknown boundary stress components directly from the approximate stress function without recourse to such indirect techniques as extrapolation, numerical differentiation, and so forth. Hence, a meaningful comparison between the specified boundary stresses and those boundary stresses consistent with the approximate biharmonic function is now made possible.

Numerous methods have been suggested for the solution of biharmonic boundary value problems, but few numerical results have appeared in the literature. Even where some results have been published, the method employed has usually been of limited application, especially dictated by the boundary conditions and geometry. Further improvement in the integral equation method, with its simplicity and generality, is likely to provide the most powerful means of attacking the biharmonic problems, thereby opening new avenues of advancement in such vital areas as elastoplasticity, viscoelasticity, second-order elasticity, and optimization.

REFERENCE

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MULTIDIMENSIONAL INTEGRALS

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A critical examination is made of a class of formulas which have been used for multidimensional integration. These formulas integrate all multivariate polynomials up to a fixed degree exactly, and the evaluation points are chosen in subspaces of low dimension in order to minimize their number.

It is shown that when the evaluation points are chosen in this way, serious errors may be introduced when the formulas are applied to general integrands.

The repeated Gauss formulas are interpreted as formulas of similar type. It is shown, using a multivariate Taylor expansion, that the Gauss formulas handle many more terms of the expansion than a multivariate formula (using an equivalent number of points) and that these terms are generally of greater importance.

It is concluded that repeated Gauss quadrature is still the best method for evaluating multidimensional integrals.

USES OF THE SUMMATION BY PARTS FORMULA IN COMPUTATION

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The summation by parts formula, probably due to Abel, has been used throughout the years in various ways. It is shown in this report how it can be used in many places to abbreviate computation. Illustrations are given for the computation of $1/n^2$, π , and e . Some of its strictly mathematical uses are pointed out; for example, a very simple proof of Euler's theorem on the convergence of infinite series. This theorem is useful in computation.

EVALUATION OF THE GEOPOTENTIAL FORCES FOR AN ORBITAL EPHEMERIS WITH GIVEN HIGH-ORDER HARMONICS

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The number of terms in the geopotential forces are rapidly becoming quite large, and care must be taken in their evaluation on the high-speed computers for the numerical integration of an orbital ephemeris. The geopotential, normally given in spherical coordinates, will suffice for discussion since it contains the basic functions and form as the geopotential forces. This potential is described in terms of zonal, tesseral, and sectoral surface harmonic terms in which the harmonic coefficients are primarily derived from satellite data. Each harmonic term consists of separable factors in spherical coordinates of geocentric latitude, longitude, and distance, including an associated Legendre polynomial referred to by degree n and order m . The individual terms are double summed over $n=0$ to N and $m=0$ to n yielding $(N+1)(N+2)/2$ total harmonic terms for the potential. It is estimated that for 1-meter satellite accuracy, a complete set of coefficients of degree $N=18$, yielding 190 terms, is required. Variation of terrestrial gravity measurements averaged over 5° square sectors of the Earth tends to support the above 1-meter figure. Intercomparisons of geodetic tracking systems, including the laser systems on the NASA GEOS-I satellite, are indicative of the need of improved gravity fields. An interesting figure, obtained by associating an individual coefficient with each 5° sector (100 000 sq. mi.) of the Earth, would require a complete potential of about $N=40$, or 800 harmonic terms, as there are two coefficients per term. The Smithsonian Astrophysical Observatory has recently derived a gravity field from combined satellite and terrestrial gravity measurements that has a complete set of harmonic coefficients up through $N=15$.

A method of evaluating the gravity forces for computer computation is presented and employs recursive formulas in spherical coordinates for accumulating the $(N+1)(N+2)/2$ harmonic terms. This is compared with a method using similar recursive formulas expressed in Cartesian coordinates, which is shown to be slightly less efficient in computer execution time. This is primarily because the potential harmonic terms are not expressible in separable factors in the Cartesian coordinates. Another method is presented where the associated Legendre polynomial in each harmonic term of the above double sum is expanded in its individual terms in the Cartesian coordinates and combined with the longitudinal factor expanded in Cartesian coordinates; this is then represented by a double sum over these combined individual terms. This representation of the potential yields a fourfold sum with an extensive kernel. It is presented, since it has been developed for use, and it demonstrates that care must be taken in designing computational schemes because it requires exorbitantly large computer time on the IBM 360. With the use of the spherical recursive formulas in evaluating the gravity forces and with the use of an 11th-order Cowell integration, it takes about 1 minute of UNIVAC 1108 computer time to generate a GEOS-I definitive orbital ephemeris over 1 day, and it is estimated that it will require less than this amount of time for a weekly orbital ephemeris on the IBM 360/95. The gravity field employed in these latter calculations and used in the GSFC orbital intercomparisons of the GEOS-I geodetic tracking systems is of degree $N=15$.

CERTIFICATION OF LIBRARY SUBROUTINES

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One of the important questions of numerical analysis is the quality of the subroutines for the approximation of elementary functions and other service chores such as exponentiation provided in a higher level language package; for example, Fortran, provided by manufacturer **XYZ** or by your own shop.

An analysis of the **IBM** System 360 library was made at the Argonne National Laboratory and published in their technical report, ANL7321.

Using slightly modified techniques, much of this work has been repeated and also carried out for much of the **IBM** 7094 Fortran version 13. The Lewis Research Center program, written in **IBM** Fortran **IV**, produces a histogram of error distribution, identifies the arguments of extreme error in each test interval, and computes the root-mean-square value of the relative error. The selection of sub-intervals and generation of test data is reasonably flexible. A source of higher precision values for each argument is required.

CLOSED-FORM SOLUTIONS FOR COAST-ARC TRANSITION MATRICES ASSOCIATED WITH OPTIMIZATION PROBLEMS

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A Cartesian coordinate, closed-form solution of the two-body problem is available from the work of W. H. Goodyear. This solution makes use of the regularizing variable ψ , such that $dt/d\psi = R$, and thus the singularities which are usually encountered at eccentricities of 0 and 1 are eliminated from the problem. Also, Goodyear develops a solution for the partial derivative matrix which indicates the effects on the coordinates of the position and velocity vector at a particular time with respect to changes in the coordinates of the initial position and velocity vector. This work has been extended to obtain a closed-form solution for the Lagrangian multipliers or adjoint variables across a coast arc and their associated transition matrices. This information can be used to construct a Newton's iteration to satisfy the boundary conditions for optimization problems and to construct perturbative numerical integration schemes for optimization problems.

A VARIATIONAL BOUNDARY METHOD FOR STRESS ANALYSIS

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A direct variational method for solving boundary value problems in stress analysis when a representation of the general solution of the field equations is known can be based on Reissner's variational equation. The method involves numerical integration over the boundary of the domain. For a problem in classical three-dimensional elasticity, the functional to be made stationary is

$$I = \iint_{S_\sigma} \left(\frac{1}{2} v_j \sigma_{ji} - p_i^0 \right) u_i dS - \iint_{S_u} \left(\frac{u_i}{2} - u_i^0 \right) v_j \sigma_{ji} dS$$

For computing I , the Papkovitch-Neuber representation of the displacement vector, u_i , in terms of four harmonic functions ϕ_i can be used. This is

$$2Gu_i = (3-4\nu) \phi_i - x_j \phi_{j,i} - \phi_{0,i}$$

For the problem of plane elasticity, the functional is similar to I . For computation, the general solution for the displacement vector in terms of two analytic functions $f(z)$ and $g(z)$ can be used. This is

$$u_i + i u_2 = \frac{3-\nu}{4} f - \frac{1+\nu}{4} z \bar{f}' + \bar{g}$$

In plate bending, the functional takes the form

$$I = \int_{C_\sigma} \left| \left(\frac{M_{nn}}{2} - M_{nn}^0 \right) \beta_n + \left(\frac{1}{2} Q_{ne} - Q_{ne}^0 \right) w \right| dS - \int_{C_u} \left| \left(\frac{\beta_n}{2} - \beta_n^0 \right) M_{nn} + \left(\frac{w}{2} - w^0 \right) Q_{ne} \right| dS$$

and can be expressed in terms of the deflection w whose general solution in terms of 2 analytic functions $f(z)$ and $h(z)$ has the form

$$w = -\frac{1}{4D} \operatorname{Re} \left(\bar{z} f - \frac{4}{1-\nu} h \right)$$

This boundary method takes advantage of the availability of the general solution of the field equations and should require less labor than interior methods, such as finite difference and finite element methods.

SOME USES AND POSSIBLE USES OF FRECHET AND OTHER DERIVATIVES IN A BANACH SPACE

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We have tried to collate a number of ideas into a cogent program. Our first acquaintance with the use of Frechet derivatives in numerical analysis was in the G. Fichera method of finding minimum L^2 error bounds in approximating solutions of boundary value problems by the least-squares continuous (i.e., not finite difference) method (refs. 1 and 2). The next was the L. V. Kantorovich theorem on convergence of the Newton method in a Banach space. This theorem and a modification of it by M. Altman were used by a student, Alan Elcrat, to prove existence (Math. Biosciences, to appear) of regular solutions of a boundary value problem related to blood flow as I had formulated it. Specification of the initial velocity profile was avoided by replacing uniqueness with mean-square asymptotic uniqueness (giving uniqueness of a rapidly acquired steady state) with the hope that this would indeed make the proof of existence of regular solutions feasible. Dr. Elcrat finds a sequence of linear boundary value problems whose solutions converge to a solution of our nonlinear boundary value problem. By mapping from $C^{2+\alpha}$ to C^α (Holder continuous functions with a Leray norm), he forces certain differential operators to be continuous, and he is then able to utilize the results of Avner Friedman's book on parabolic type to obtain existence for the linear problems. We now plan to approximate solutions of the linear problems using least squares, accumulating error bounds of these and the relevant Newton method to obtain an overall bound for the error, but we intend to try for the fastest possible convergence properties before we start computing. Toward this end, we are examining asymptotic stability of the analogs

$$dx/dt = -[P'(x)]^{-1}P(x) \quad \text{or} \quad dx/dt = -[P'(x_0)]^{-1}P(x)$$

of the Newton method and modified Newton method, where $P:A \subset X \rightarrow Y$ and X, Y are Banach spaces and $P'(x)$ is the Frechet derivative of P . We may possibly integrate such equations numerically using Runge-Kutta in order to find an iterative method which is faster than the Newton method. Work in a forthcoming textbook (Harper & Row) on

$$dx/dt = -(I+B^{-1}C)x + B^{-1}b ,$$

an analog for the general linear iterative methods

$$x^{k+1} = -B^{-1}Cx^k + B^{-1}b ,$$

for linear systems

$$(B+C)x = Ax = b ,$$

which is gotten from it by a crude Euler first-order finite difference replacement, is encouraging in this direction. Here the analog is asymptotically stable if the iterative method converges, and its asymptotes

are solutions of the linear system. The converse is not true. A Runge-Kutta method should provide quicker access to the asymptotes than the Euler method.

It was suggested that further work is intended on our simultaneous replacements and accelerated successive replacements iterative methods for nonlinear systems (ref. 3) and on another method which should almost converge unconditionally and derives from nonlinear least-squares considerations.

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SOLUTION OF LINEAR SYSTEMS OF VERY LARGE SIZE

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The subject of how to solve linear systems of very large size (high rank) is too large for a short discussion. A particular method of solution having outstanding peculiarities that are theoretically important will therefore be discussed. The method, invented by Michael Creusen (formerly of Autometric/Raytheon) is characterized by the following unusual features.

(1) The customary system to be solved has the form

$$\begin{matrix} Y \\ n \times 1 \end{matrix} = \begin{matrix} A \\ n \times m \end{matrix} \begin{matrix} X \\ m \times 1 \end{matrix} \quad m \leq n$$

Creusen considers instead the system

$$\begin{bmatrix} 0 \\ n \times 1 \end{bmatrix} = \begin{matrix} B \\ n \times m \end{matrix} \begin{matrix} Z \\ m \times 1 \end{matrix} \quad m \geq n$$

which is made equivalent to the former.

(2) If we write for the variances:

$$\begin{bmatrix} \sum \bar{z}\bar{z} & \sum \bar{z}w \\ \sum w\bar{z} & \sum ww \end{bmatrix} = B \sum \bar{z}\bar{z} B^T$$

where

$$\begin{aligned} W &\equiv B \bar{Z} \\ -WC &= \Delta Z \\ Z &= \bar{Z} + \Delta Z \end{aligned}$$

and

$$[I] = [B_0 B_1] \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}$$

then

$$\Delta Z = \{B_0^{-1} (I - B_1 C_1)\} (B_0 B_1) \bar{Z}$$

Now we find a unique value **for** C by minimizing, not as usual the scatter in $\underline{\Delta Y}$ but instead the scatter in $\underline{\Delta}$. This gives

$$C = \begin{bmatrix} 0 & I \end{bmatrix} \left[\sum_{\bar{z}\bar{w}} \sum_{ww}^{-1} \right]$$

so that

$$\Delta Z = -Z_{z\bar{w}} Z_{ww}^{-1} W$$

(3) It can be shown that the Creusen procedure always converges to a solution, even when A is singular **or** near singular.

(4) Writing the problem so that $m \geq n$ allows great flexibility in programing the equations by minimizing storage space and by allowing sequential sdlution in steps of any size.

The procedure has, on the other hand, certain disadvantages. These have not been fully worked out yet.

HEURISTIC INTEGRATION PROGRAM

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This discussion reports on a heuristic program for the numerical integration of systems of ordinary differential equations.

The computer program was designed to incorporate the following features: (1) to meet a user specified accuracy; (2) to be problem independent (i.e., any number or degree of coupled differential equations); (3) to be self-optimizing with respect to step size, order, and integration methods used; and (4) to exhibit learning so as to use a past-performance history to determine methods and orders to be used.

The integration methods used are: (1) the method of Adams, Bashforth, and Moulton; (2) Butcher's formulas for the Stetter-Gragg-Butcher method; (3) Cowell's method of nth-order differences; and (4) Shank's formulas for the Runge-Kutta method.

The programs described here are written in both single (11-decimal place) and double (22-decimal place) precision Algol for the B 5500. An executive procedure acts in an administrative and bookkeeping capacity for the various integration methods. This executive procedure keeps the performance histories according to problem types, determines performance effectiveness of the methods and orders, and chooses those to be used.

Several kinds of problems and a large range of accuracies were used to exercise the program with the following results:

- (1) All methods perform well with no method or order always being exceptionally superior to the others
- (2) For different kinds of problems and accuracies different methods and orders prove more effective
- (3) Learning takes place in a satisfactory manner
- (4) The program makes an effective general library procedure for integrating systems of ordinary differential equations.

REFERENCE

Gallaher, L. J.; et al.: Study of the Methods for the Numerical Solution of Ordinary Differential Equations. Final Report Project A-831, Contract NAS8-20014, Engineering Experiment Station, Georgia Institute of Technology, Atlanta, Ga., NASA CR-82962, 1967.

NUMERICAL TECHNIQUES IN ORBITAL LIFETIME PREDICTIONS

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Since the necessity of integrating over long periods of time, when one is calculating orbital lifetime, the classical methods of trajectory calculations cannot be used. The rate of change of apogee and perigee is a slowly varying quantity, when compared to the change in some sort of position vector. Hence, the rate of change of apogee and perigee can be integrated for long periods of time while keeping error down to a minimum.

A Simpson's rule of integration is used to calculate values of $\dot{\mathbf{A}}$ and $\dot{\mathbf{P}}$, which are orbit and vehicle dependent. These values are then integrated in time by a fourth-order Runge-Kutta scheme.

Accuracy is within the state of art of the environmental modeling, and orbital ephemerides as long as 10 000 days can be calculated in a matter of minutes.